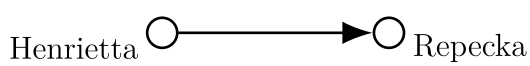
King Chicken Theorems

"Mathematics makes the mind its playground"

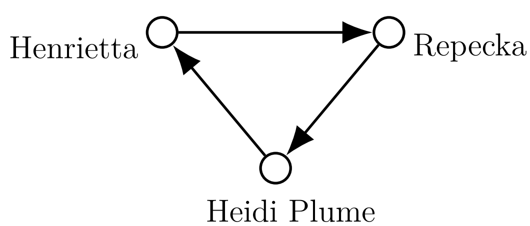
-Francis Su, Former President of the Mathematical Association of America

THE QUESTION: In a flock of chickens, for every two chickens in the flock, exactly one chicken pecks the other.
However, it is not clear who is the **head chicken** in the flock. Perhaps Henrietta pecks Repecka and Repecka pecks Heidi Plume. But if Heidi Plume pecks Henrietta, who's really in charge here?

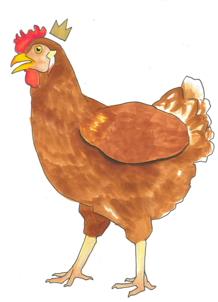
THE MATH: It's convenient to represent a flock of chickens with a **digraph**, which is a collection of **vertices** and **directed edges**. The chickens become vertices, and we draw an edge from Henrietta to Repecka if Henrietta pecks Repecka:



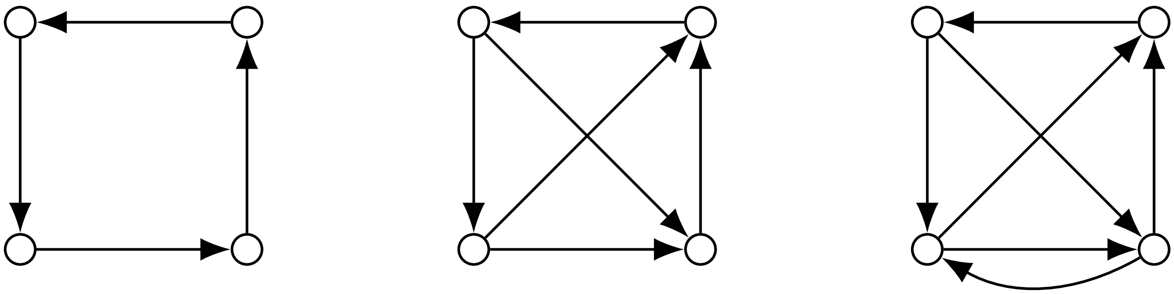
So the scenario where Henrietta pecks Repecka, and Repecka pecks Heidi Plume, and Heidi Plume pecks Henrietta, would be represented as:



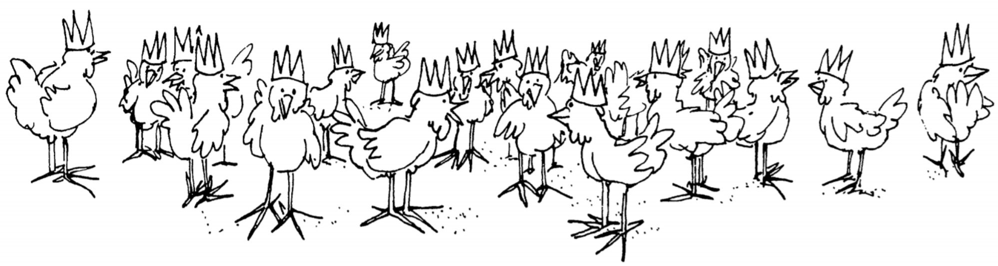
We define a **King Chicken** as a chicken c, so that for every other chicken, d, either c pecks d, or c pecks some third chicken, b, and b pecks d.



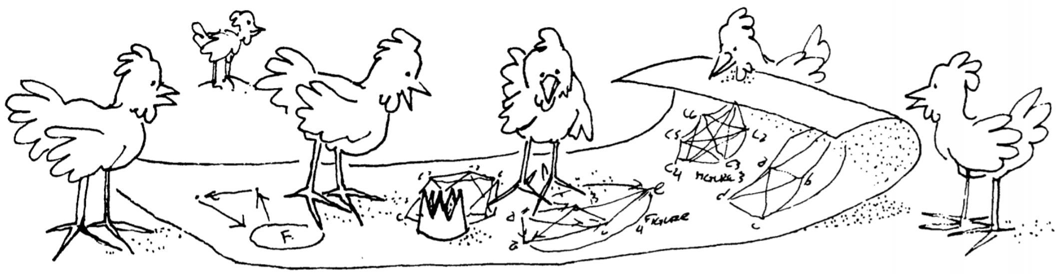
1. Which of the following is a valid flock of chickens, according to our rule that for every two chickens in the flock, exactly one chicken pecks the other?



2. Can you make a flock of chickens where **every chicken is a king?**



3. Can you make a flock of chickens where **exactly one chicken is a king?**

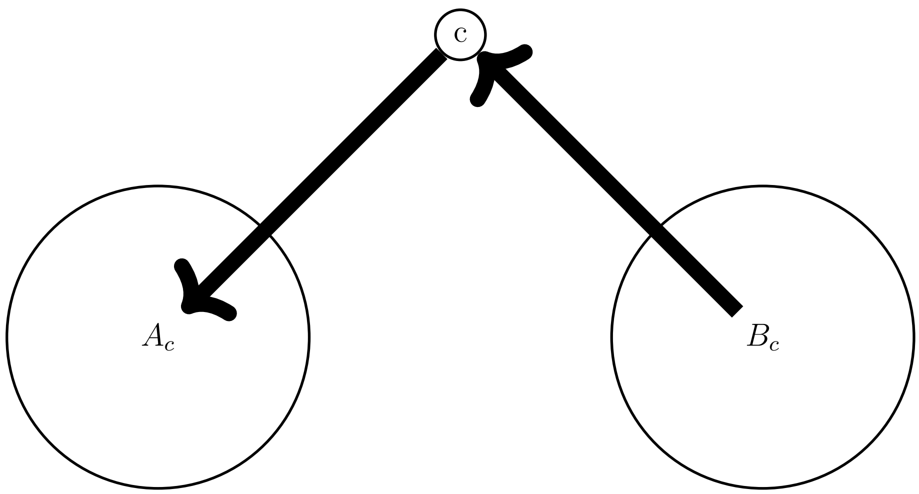


4. Can you make a flock of chickens where **no chicken is a king?**

5. Some Notation

- Let A_c be the set of all chickens that chicken c pecks.
- Let B_c be the set of all chickens that peck chicken c.

From now on, a big circle will represent a set of vertices, and a thick arrow from a big circle to a vertex will mean every edge between the set and the vertex goes in the direction of the big arrow. So:



6. Proof that **Every flock of chickens has a king!**

- Let $a(c)$ be the number of chickens that chicken c pecks.
- Let c be the chicken for which $a(c)$ is maximum.

We will show that c is a king by a **proof by contradiction!**
Suppose c is **not** a king. Then there's some other chicken, d, that c doesn't peck, and no chicken that c does peck pecks chicken d.
How many chickens does d peck ?

Chicken d pecks chicken c, and also, chicken d pecks every chicken in A_c. But then chicken d pecks at least $a(c) + 1$ chickens! **This is a contradiction!**

7. We will now show the following lemma (helper fact):

   Lemma 1 *Every chicken who is pecked is pecked by a king.*

   - Suppose chicken c is pecked; then the set B_c has at least one chicken in it.
   - Consider the set B_c as a flock by itself; that is, ignore all the other chickens and connections to those other chickens.
   - We know by the theorem we just proved that the subflock, B_c, has a king.

   I claim that the king of B_c is the king of the entire flock. Why?

8. If a flock of chickens has exactly one king, then ______________________________

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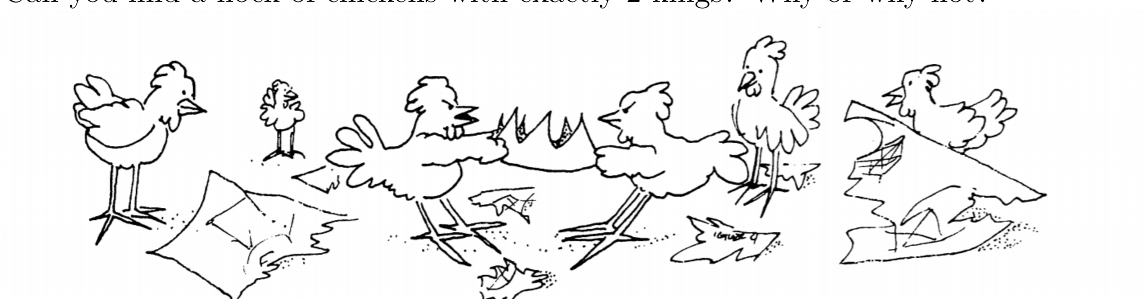
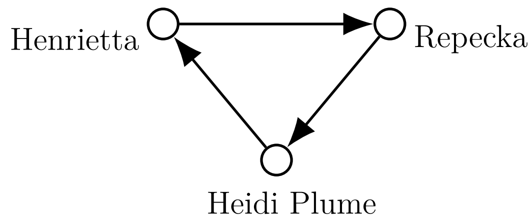
9. Can you find a flock of chickens with exactly 2 kings? Why or why not?



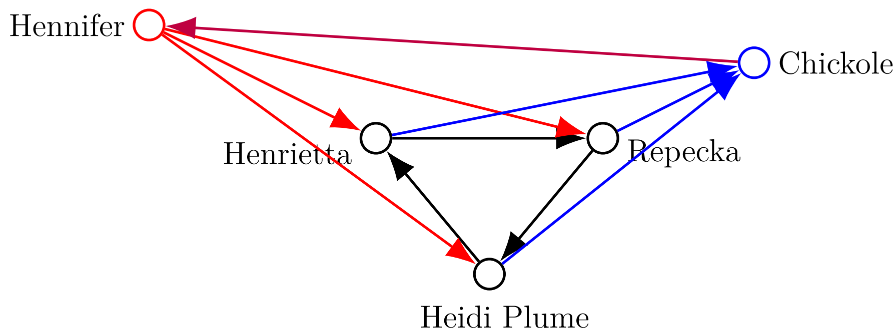
10. Now we want to tackle the question: “When can you find a flock where **every chicken is king**?”

    We already found a flock of 3 chickens where every chicken is king:



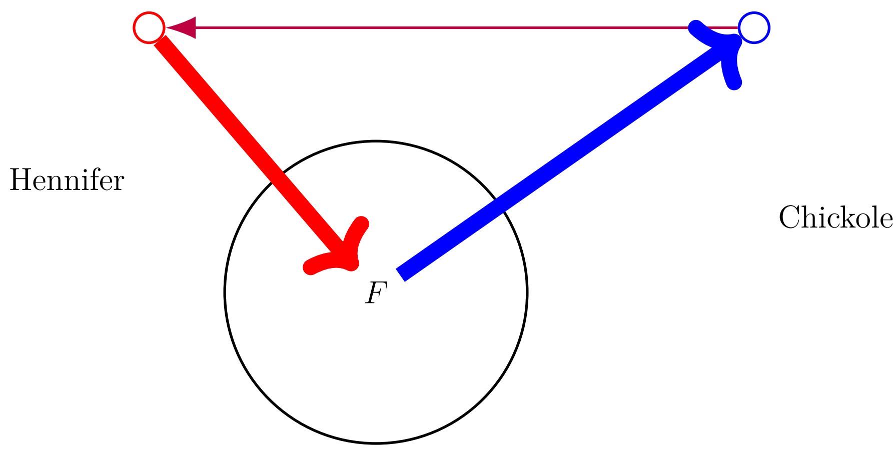
    Can you find a way to add two new chickens, Chickole and Henelope, to the existing flock, so that you get a flock of 5 chickens, all of which are kings?

Suppose I add two new chickens: Chickole and Henelope, in the following way:



Are all 5 chickens kings?

Suppose I have a flock, F, of n chickens where all n chickens are kings, and I add two chickens in the same way I did above:



Are all chickens kings in the new flock of $n+2$ chickens? Why or why not?

We've shown there exists a flock of n chickens, all kings, when ________________________

Proof that there is no flock of four chickens, all kings

Suppose we had a flock of four chickens, all kings. Determine how many chickens each chicken must peck (i.e., no chicken pecks three other chickens- why? Can all four chickens peck two other chickens?).

Sketch the graph(s) you have just determined.

Can you find a flock of 6 chickens, all kings?

11. Suppose we create a **random n-flock** of chickens by randomly assigning directions of the edges. So for every pair of chickens, c and d, assign the event that c pecks d probability $\frac{1}{2}$ and assign the event that d pecks c probability $\frac{1}{2}$.

Some Probability Facts

- If E is an event with probability $P(E)$, then the probability that E **doesn't** occur is $1 - P(E)$.
- If E and F are two events which **cannot occur simultaneously**, then the probability that either of E or F occurs is $P(E) + P(F)$.
- If E and F are two events which **can occur simultaneously**, then the probability that either of E or F occurs is **less than** $P(E) + P(F)$.
- If E and F are **two independent events** (that is, E and F have no effect on each other), then the probability that both E and F occur is $P(E)P(F)$.

We will use the following fact of probability. Let $\mathrm{P}(E)$ denote the probability that an event E occurs. Let $\{E_i\}$ be a set of events, and let $\bigcup E_i$ be the event that at least one of the E_i's occurs. Then

$$P(\text{at least one of the } E_i\text{'s occurs }) \leq P(E_1) + P(E_2) + \cdots + P(E_m)$$

or, in more Math-y terms,

$$P\left(\bigcup E_i\right) \leq \sum P(E_i)$$

with equality if and only if the E_i are mutually exclusive (that is, if E_1 happens, then E_2 definitely can't happen).

(a) What is the probability that, in a random n-flock, Henrietta pecks Repecka **and** Henrietta pecks Heidi Plume?

(b) What is the probability that, in a random n-flock, Henrietta pecks every other chicken?

(c) What is the probability that, in a random n-flock, Henrietta pecks every other chicken, **OR** Repecka pecks every other chicken?

(d) What is the probability that a random n-flock has exactly one king?

(e) What happens to the probability that a random n-flock has exactly one king as n becomes very large?

12. Now suppose we want to know the probability that a random n-flock has all kings. I'm not sure how to do this! Instead, we'll develop an estimate for the probability that **not** every chicken is king.

- Let $p(n)$ denote the probability all n chickens are kings.
- Let $\overline{p(n)} = 1 - p(n)$ denote the probability that not all chickens are kings.
- We will **overestimate** $\overline{p(n)}$ and thus **underestimate** $p(n)$.

(a) If not every chicken is king, there is some pair of chickens, c and d, such that c does not peck d and no chicken that c pecks also pecks d. Let $p(c, d)$ denote the probability that some particular pair of chickens has this property. How many such ordered pairs are there in a flock of n chickens?

By symmetry, all the $p(c, d)$ are equal for every pair of chickens. Thus by the probability fact in question 10, $\overline{p(n)} \leq$ ________________ $p(c, d)$

(b) Now we will calculate $p(c, d)$. What is the probability that c does not peck d?

(c) For a **specific** third chicken, b, what is the probability that c does not peck d **through** b? (i.e., either c does not peck b, or b does not peck d, or both?)

(d) Who pecks who among c and d is independent of their relationship with any third chicken, b. And their mutual relationship with any one b is independent of their relationship to any other b. So

$$p(c, d) = \rule{6cm}{0.4pt}$$

Using our estimate from part (a), we have

$$\overline{p(n)} \leq \rule{8cm}{0.4pt}$$

(e) What happens to our overestimate from part (d) as n gets very large?

(f) What can you conclude about the probability that a random n-flock has all kings, as n becomes very large?

Questions For You To Ponder!

1. For any pair of whole numbers, n and k with $k \neq 2$ and $k \leq n$, can you find a flock of n chickens with k kings?
2. Suppose I call a chicken, c, a **serf** if every chicken either pecks it, or pecks some chicken who pecks c. Can you find a flock of all serfs? No serfs? Exactly 2 serfs?
3. If a flock is all kings, how many serfs are there?
4. There can be lots of king chickens- perhaps we need a more restrictive definition! Suppose instead, we think of the **king of kings** chickens: first, find all the kings in a flock. Then, think of this as a subflock of the entire flock, and find the king of this smaller flock. When will this be the same? When will this be different?
5. Suppose I want to not only find the king of the chickens, but rank every chicken from most to least powerful. How might one do this?

For more information, see "The King Chicken Theorems" by Stephen B. Maurer (easily google-able). For questions, comments, and/or a link to the article, email me at: jkenkel@umich.edu

All illustrations in this worksheet come from the original paper by Stephen Maurer, except the chicken in the crown on the first page, which was drawn by me, Jenny.