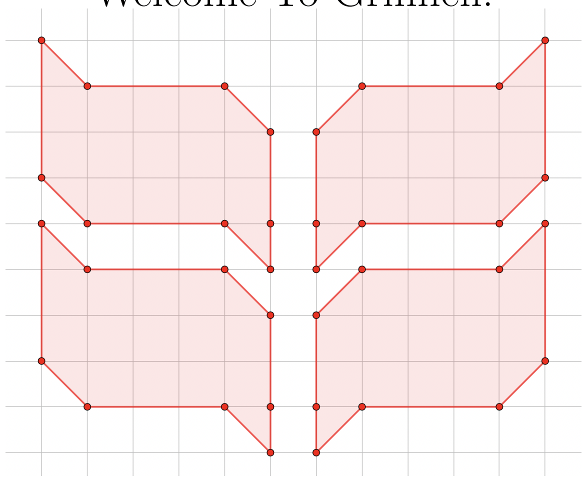
Welcome To Grinnell!

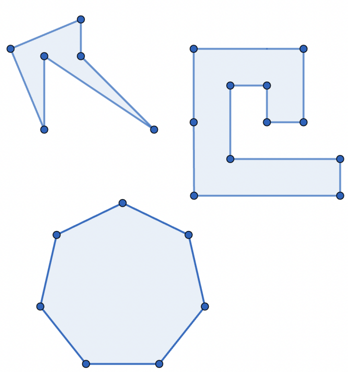


An **integer** is a whole number, either positive or negative.

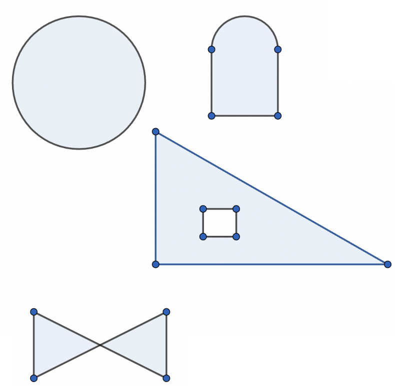
Examples: $\ldots, -2, -1, 0, 1, 2, \ldots$ **Non-examples:** $\frac{1}{2}, \pi, \sqrt{2}, -100.1, \ldots$

We aim to find the area of a polygon with vertices on integer points. A **polygon** is a closed shape made of straight lines that don't intersect, with no holes. The edges and angles do not have to be the same size.

Examples of polygons:



Nonexamples of polygons:



Points on the xy-plane where both coordinates are integers are called, uncreatively, **integer points**. For example, points like $(0,1),(2,-5)$ or $(-20,-24)$ are integer points, while $(\frac{1}{2},2)$ is not. Integer points are some times called **lattice points**, because they look like they make a lattice on the xy-plane.

Figure 1: From homedepot.com, "8 ft. x 4 ft. Pressure Treated Wood Pine Square Privacy Lattice"

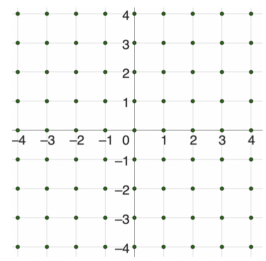
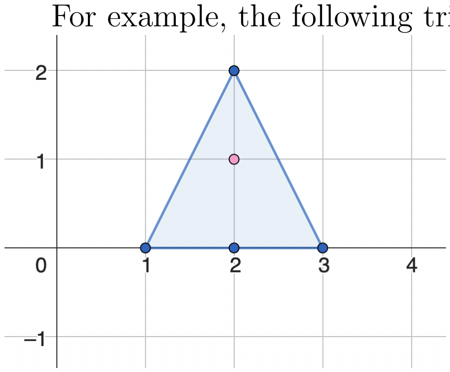


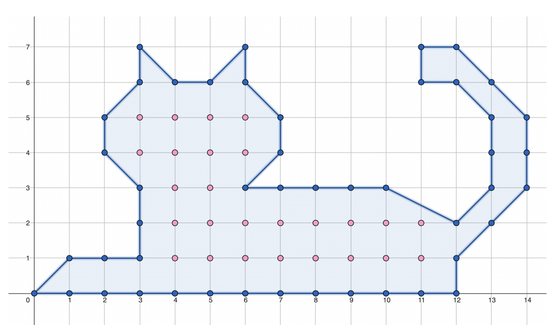
Figure 2: The set of integer points in the xy-plane

THE QUESTION: Suppose all corners of a polygon are at **integer points**. Can we figure out the **area** of the polygon by counting the number of integer points that occur in the polygon?

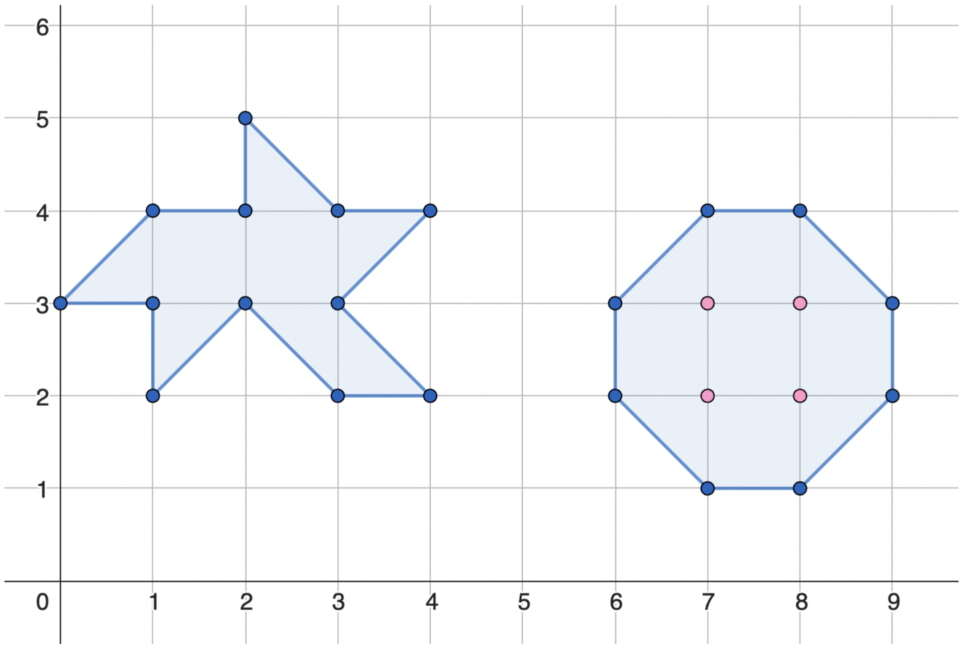
For example, the following triangle contains 5 integer points:



Can we figure out the area of this polygon without having to do a bunch of nasty calculations?



1. It turns out that just counting integer points is not quite enough. Find the area[1] of both the polygons below and count the number of integer points they each contain.



Number of integer points
in the polygon: ____________
Area of the polygon: :____________

Number of integer points
in the polygon:____________
Area of the polygon: :____________

2. Keeping the example above in mind, explain why it would be impossible to figure out the area of a polygon by only counting the number of integer points that occur in the polygon.

Notice that the polygons in question 1 have different numbers of **interior** integer points and **boundary** integer points, so maybe we can use *that* data to figure out the area of a polygon.

[1]Recall: the area of a triangle is $\frac{1}{2}\times$base$\times$height, where the base and height are perpindicular to each other.

3. Find the area, number of interior integer points, and number of boundary integer points for a bunch of polygons. There are some polygons provided, but you can also draw your own (keeping in mind that you want to calculate their areas, so they shouldn't be *too* weird). Record your data in the table below.

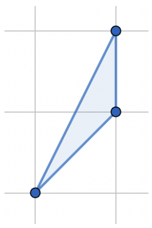
   Form a conjecture (math-word for "guess") about the relationship between the area of a polygon, the number of interior integer points, and the number of boundary grid points.

Area	Number of interior integer points	Number of boundary integer points

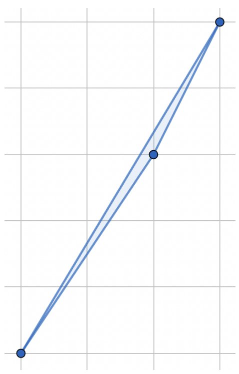
Let P be a polygon with vertices that are integer points, let I be the number of interior integer points of P, and let B be the number of boundary integer points of B. Pick's Theorem says that the area of any polygon with vertices on integral points is given by

Area = ________________________________

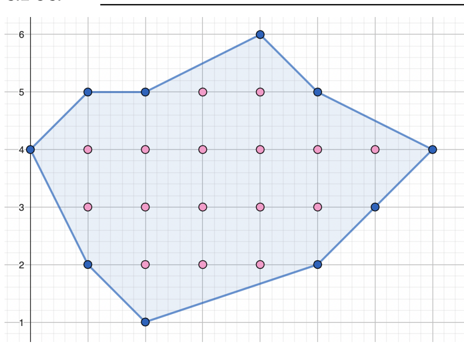
4. Use Pick's theorem to find the areas of the following polygons.

(a) 

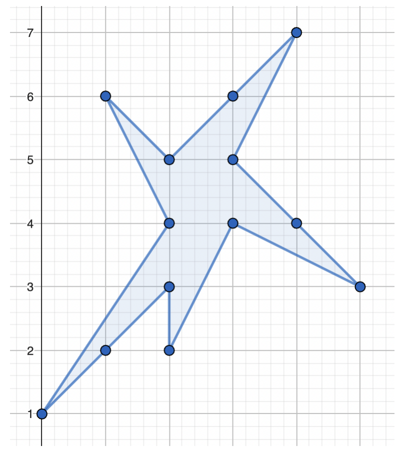
area = ____________________________

(b) 

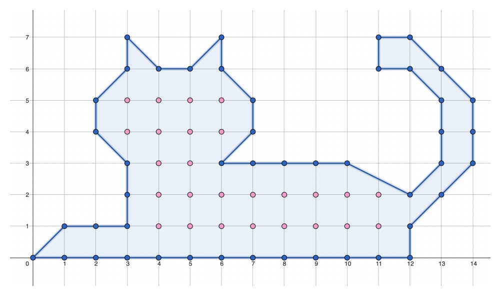
area = ____________________________

(c) 

area = ____________________________

(d) 

area = ____________________________

(e) 

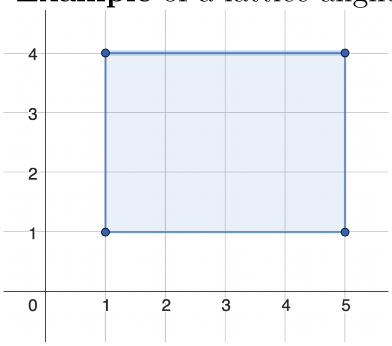
area = ____________________________

5. Draw a weird shape with vertices at integer points and swap your paper with someone else. Have them find the area of your shape.

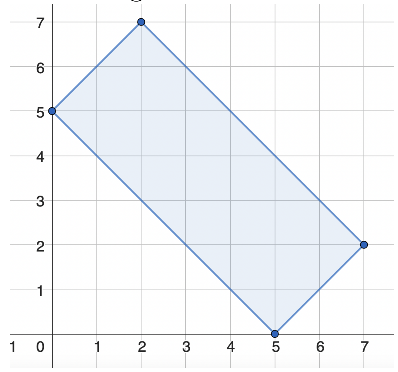
But how can you be sure this will be true for every single polygon, without drawing them all? **Proving** statements are true is what mathematicians do all day!

6. In this part, we will prove that Pick's theorem is true for lattice-aligned rectangles.

Example of a lattice aligned rectangle:



A rectangle that is **not** lattice aligned:



This is a common proof technique: prove a fact for simple cases, and see if you can use those cases to prove it in more generality.

Suppose a rectangle is aligned with the integer lattice, has corners on integers, and has length m and width n.

(a) What is the area of the rectangle?

(b) What is the number of boundary points of the rectangle?

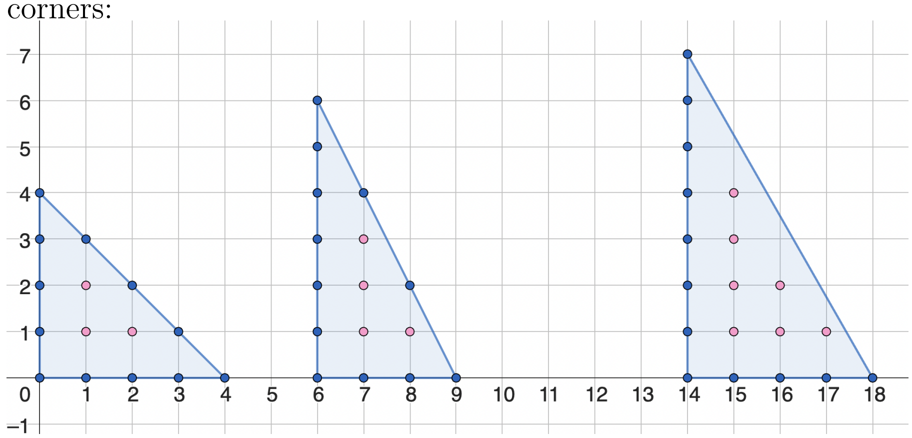
(c) What is the number of interior points of the rectangle?

(d) Prove that Pick's theorem is true for lattice-aligned rectangles. (Hint: start with an expression for $I + \frac{B}{2} - 1$, using the previous parts).

7. Now we will prove Pick's theorem for lattice aligned right triangles. Suppose a triangle has legs of length m and n.

   (a) Count the number of lattice points along the legs of the triangle. Check your answer with some of the drawings you made previously.

   (b) The number of lattice points along the hypoteneuse of a triangle is harder to count; the hypoteneuse might hit lots of integer points, or some, or none other than the corners:

   Call the number of lattice points along the hypoteneuse, not counting the corners of the triangle, k. For example, in the picture above, for the leftmost triangle k is 3, for the middle triangle k is 2, and for the rightmost triangle, k is 0.

   If the triangle is m by n and there are k lattice points on the diagonal, how many total **boundary** integer points are there for a lattice aligned right triangles? Your answer should be in terms of m, n and k:

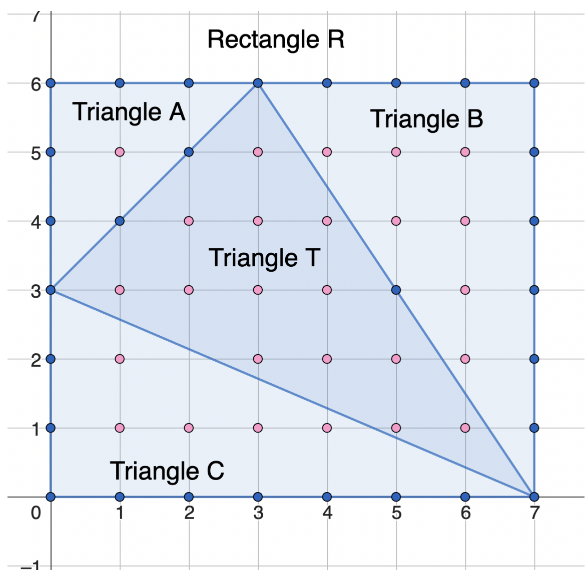
   ____

   (c) If the triangle is m by n and there are k lattice points on the diagonal, how many total **interior** integer points are there for a lattice aligned right triangles? Your answer should be in terms of m, n and k: __________________________________ Turn the page upside down for a hint. [2]

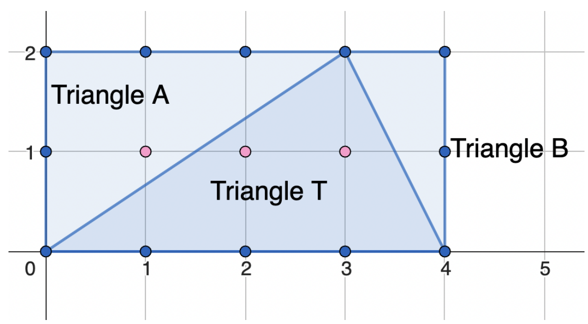
[2] Hint: Consider the rectangle with width m and length n formed by the legs of the triangle. We already figured out how many internal integer points this shape has. Some of those will live on the hypoteneuse. Half of what remains will be in the interior of the triangle.

(d) Prove that Pick's theorem is true for lattice-aligned right triangles.

8. Now we will prove Pick's theorem for **any** triangle. There are a couple of cases to consider, but basically all the cases look like one of these examples:



Case 1



Case 2

In the above pictures, we know that Pick's theorem is true for the rectangle and triangles A, B and C, but we haven't proven it for triangle T.

We will prove the theorem for the first case, but try the other case on your own!

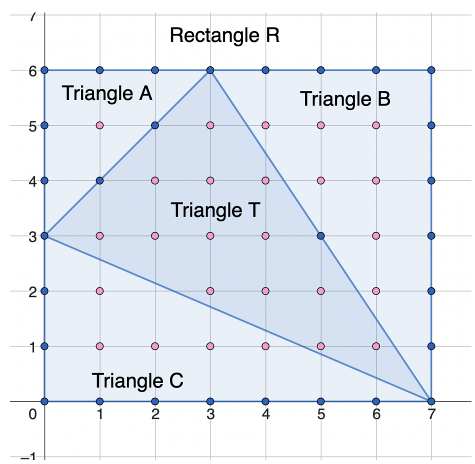
9. Suppose T is an arbitrary triangle (not the specific one above), and we form a rectangle similar to in the diagram. Let's use the following notation:
Call the rectangle formed by all four triangles R. Let

$$I_t = \text{ number of interior points in triangle } T$$
$$B_t = \text{ number of boundary points in triangle } T$$
$$A_t = \text{ area of triangle } T$$

and similarly, I_a is the number of interior points in triangle A, I_r be the number of interior points of the rectangle, and so on.

(a) Find an equation that describes how A_t is related to A_r, A_a, A_b and A_c.
$A_t =$ ____

(b) Find an equation that describes B_r in terms of B_a, B_b, B_c and B_t. Be careful of overcounting.
$B_r =$ _________________________



Check your answer for the specific example:
B_r in the specific example = ____

(c) Find an equation that describes I_r in terms of I_a, I_b, I_c, I_t and B_t. Be careful of the corners of T!
$I_r =$ ____
Check your answer for the specific example above.
I_r in the specific example = ____

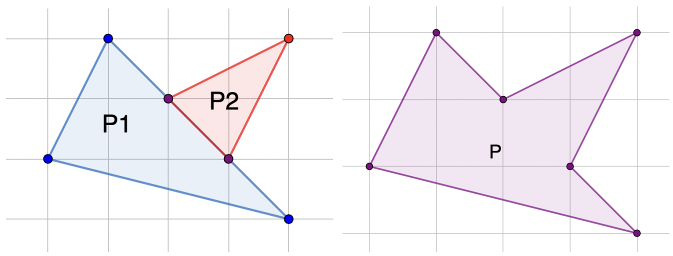
(d) Use the fact that we have proven that Pick's theorem is true for triangles A, B, C and rectangle R and your answer to the previous problem to find an equation for A_t in terms of $I_r, I_a, I_b, I_c, B_r, B_a, B_b$ and B_c. Your equation can have other constants in it.

$A_t =$ ____

(e) Use the fact that we have proven that Pick's theorem is true for triangles A, B, C and rectangle R, and your answers to the previous questions, prove that Pick's theorem is true for arbitrary triangles (at least in case 1).

10. We have shown that Pick's theorem is true for any triangle. Now we will prove that if Pick's theorem is true for two polygons, P_1 and P_2, then Pick's theorem is true for the bigger polygon P formed by "joining up" P_1 and P_2.

For example, in the diagram below, we know that Pick's theorem is true for the two triangles. This will allow us to show that Pick's theorem is true for the weird hexagon formed by smushing the two triangles together.



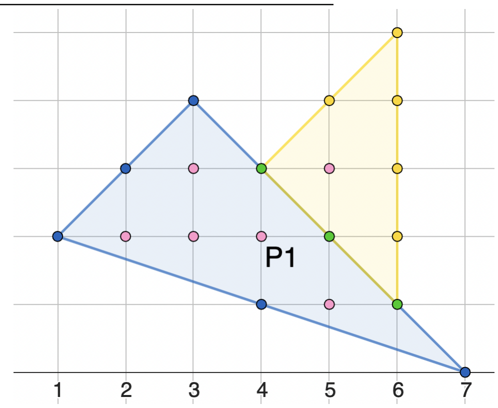
(a) Call I_{P_1} the number of interior points in polygon 1, and so on. Assume that the common line shared by P_1 and P_2 contains m integer points. Call I the number of interior integer points in P, B the number of boundary points of P, and A the area of P.
Give an expression for A in terms of A_{P_1} and A_{P_2}.
$A =$ ______________________

(b) Recall that, by assumption, Pick's theorem is true for P_1 and P_2. Give an expression for A in terms of $I_{P_1}, I_{P_2}, B_{P_1}$ and B_{P_2}.

$A =$ ____

(c) Any interior integer point of P is an interior point of P_1 or P_2 or is on the edge shared between them. Find an expression for I in terms of I_{P_1}, I_{P_2} and m.

$I =$ ____

Check your answer on this example: 

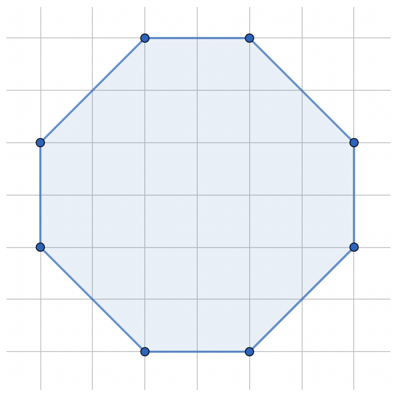
(d) Find an expression for B in terms of B_{P_1}, B_{P_2} and m.

$B =$ ____

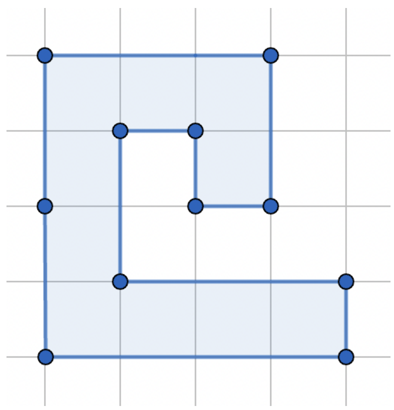
Check your answer on the example above.

(e) Use the previous parts to prove Pick's theorem for P.

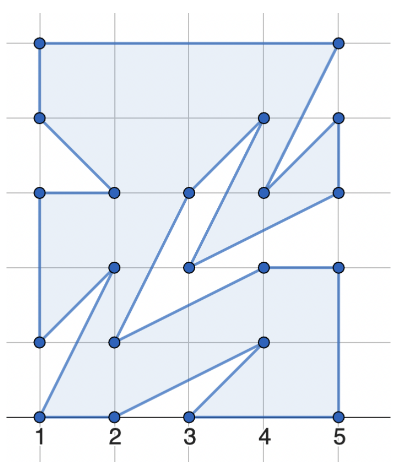
11. The last part of the proof of Pick's theorem is to show that any polygon with integer vertices can be broken up into triangles with integer coordinates. We won't prove this part (☹). Convince yourself it is true by breaking up each of the following polygons into triangles with integer coordinates.



(a)



(b)



(c)

12. Challenge somebody else: draw a polygon and swap your paper with someone else to make them triangulate it. Remember that the triangles must have vertices on integer points!

We did it!!

Thank you for your hard work!

If you have any questions, don't hesitate to reach out at kenkeljennifer@grinnell.edu!

I drew a lot of inspiration from notes on Pick's theorem by Tom Davis which can be found at `https://mathcircle.berkeley.edu/sites/default/files/archivedocs/2012_2013/lectures/1213lecturespdf/BMC_Int_Sept4_2012_Picks.pdf`.

The outdoor activity we did is called "Jumping Julia" and came from the Julia Robinson Mathematics Festival. See their website here:
`https://jrmf.org/puzzle/jumping-julia/`,